Adelic Topology

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May 7, 2024

Let k be a number field, $\mathbb{A}$ its adele ring and G a connected reductive group over k.

1 The Topology of Local Fields

Given a global field and a place ν we get a local field k_ν an an induced metric topology. It has the properties:

- All are locally compact
- For all but finitely many ν, k_ν is compact.
- k_ν, k_ν^* are both totally disconnected

2 The Topology on $\mathbb{A}$

We have defined the adeles as a topological product, hence the topology is already specified here we give only properties of this topology. Recall that we identify k with its image in $\mathbb{A}$ under the inclusion map

$$\alpha \mapsto (\alpha)_\nu$$

i.e. the constant sequence. With this in mind the adelic topology has the properties:

- Locally compact and Hausdorff
- k is a discrete set in $\mathbb{A}$
- k is closed in $\mathbb{A}$
- $\mathbb{A}/k$ is compact in the quotient topology
- For any finite set of places S k is dense in

  $$\prod_{\nu \notin S}{}' k_\nu$$

  the removal of just one place takes us from discrete to dense.

3 The Topology on $G(\mathbb{Q})$

Throughout X, Y, Z are affine schemes of finite type over R where R is a topological ring.

We first identify $X \cong \mathrm{Spec}(R[t_1, ..., t_n]/I)$ then $X(R)$ is naturally the set of points in R^n on which the polynomials in I all vanish. We then give it the subspace topology.

Theorem. *This is the unique topology on the R points of X, Y, Z that is*

- *Functorial: If $X \to Y$ is a morphism of schemes over R then $X(R) \to Y(R)$ is continuous*
- *Compatible with pullbacks: $(X \times_Y Z)(R) \cong X(R) \times_{Y(R)} Z(R)$; homeomorphic topological spaces.*
- *Compatible with embeddings: A closed immersion $X \hookrightarrow Y$ is sent to a topological embedding $X(R) \hookrightarrow Y(R)$*
- *Compatible with R points:* $\mathrm{Spec}(R[t])(R) \cong R$ *topologically.*

Moreover under this topology $X(R)$ forms a topological group, and if R is locally compact, or Hausdorff so is $X(R)$.

4 The Topology on $G(\mathbb{A})$

[Con12] is the reference. Let X be an affine scheme of finite type over a global field F. Then $X \cong \mathrm{Spec}(F[t_1, ..., t_n]/I)$ and $X(F)$ is naturally the set of points in F^n on which the polynomials in I all vanish. $X(\mathbb{A})$ is identified with the subset of $\mathbb{A}^n$ where $f : \mathbb{A}^n \to \mathbb{A}$ in I vanish. We give this the subspace topology.

References

[Con12] Brian Conrad. Weil and Grothendieck approaches to adelic points. *L'Enseignement Mathématique*, 58(1):61–97, June 2012.